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The massless representations of the conformal quantum algebra

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Abstract. The massless representations of the conformal quantum algebra $\mathcal{U}_q(su(2,2))$ for complex q such that $|q|=1$ are studied in detail. By factorizing out all singular vectors of the corresponding Verma modules, a simple basis for these representations is explicitly constructed. This basis is new also for the usual conformal algebra $su(2,2)$. This construction allows a straightforward treatment of the case q a root of unity, when the representations are unitary and finite-dimensional.

1. Introduction

The conformal quantum algebra $\mathcal{U}_q(su(2,2))$ is a q-deformation of the ordinary Lie algebra $su(2,2)$ [1]. Some results on the study of its irreducible representations were presented in [2]. In particular, for generic q, such that $|q|=1$, the representations with positive energy are deformations of the corresponding representations of $su(2,2)$. When the deformation parameter q is a root of unity, the picture of the representations changes drastically and all positive-energy irreducible representations are unitary and finite-dimensional.

In the present paper we continue the study of the representations of $\mathcal{U}_q(su(2,2))$, by examining in detail the case of the massless representations. For all q such that $|q|=1$, we construct explicitly a simple basis for the corresponding representation space. This basis is new also for the usual conformal algebra $su(2,2)$. It is built by an explicit two-step factorization of all singular vectors of the corresponding Verma module over $\mathcal{U}_q(sl(4,\mathbb{C}))$ together with the appropriate reality condition. In this basis it is manifest that each weight of a massless representation has multiplicity one. Notice that these representations are pseudo-unitary and become unitary in the limit $q\to 1$. For q a root of unity, the basis truncates and the representation space becomes finite-dimensional. For $q=\mathrm{e}^{2\pi i/N}$, $N=2,3,\ldots$, all basis vectors have now positive norm, and the representations become unitary. Our results are in the most general form since the representations are obtained using factor-modules of Verma modules.

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2. Preliminaries

The quantum algebra $\mathcal{U}_q(sl(4,\mathbb{C}))$ is defined as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^{\pm}$, H_j, $j=1,2,3$, and with relations [3,4]:

$$[H_j, H_k]=0 \qquad [H_j,\ X_k^{\pm}]=\pm a_{jk}X_k^{\pm} \qquad [X_j^{+}, X_k^{-}]=\delta_{jk}[H_j]_q \tag{1a}$$

$$\left(X_j^{\pm}\right)^2 X_k^{\pm} - [2]_q X_j^{\pm} X_k^{\pm} X_j^{\pm} + X_k^{\pm}\left(X_j^{\pm}\right)^2 = 0 \qquad (jk)=(12),(21),(23),(32)$$
$$[X_1^{\pm}, X_3^{\pm}]=0 \tag{1b}$$

where $[x]_q=\left(q^{x/2}-q^{-x/2}\right)/\tilde{\lambda}$, $\tilde{\lambda}\equiv q^{1/2}-q^{-1/2}$, $(a_{jk})=(2(\alpha_j,\alpha_k)/(\alpha_j,\alpha_j))$, $j,k=1,2,3$, is the Cartan matrix of $sl(4,\mathbb{C})$; $\alpha_1,\alpha_2,\alpha_3$ are the simple roots; the non-zero products of the simple roots are: $(\alpha_j,\alpha_j)=2$, $j=1,2,3$, $(\alpha_1,\alpha_2)=(\alpha_2,\alpha_3)=-1$. The non-simple positive roots are $\alpha_{12}=\alpha_1+\alpha_2$, $\alpha_{23}=\alpha_2+\alpha_3$, $\alpha_{13}=\alpha_1+\alpha_2+\alpha_3$. The elements H_j span the Cartan subalgebra $\mathcal{H}$, while the elements $X_j^{\pm}$ generate the subalgebras $\mathcal{G}^{\pm}$.

The Cartan–Weyl basis for the non-simple roots is given by [4,5,1]:

$$X_{jk}^{\pm}=\pm q^{\mp 1/4}\left(q^{1/4}X_j^{\pm}X_k^{\pm}-q^{-1/4}X_k^{\pm}X_j^{\pm}\right) \qquad (jk)=(12),(23) \tag{2a}$$

$$\begin{aligned} X_{13}^{\pm} &= \pm q^{\mp 1/4}\left(q^{1/4}X_1^{\pm}X_{23}^{\pm}-q^{-1/4}X_{23}^{\pm}X_1^{\pm}\right)\\ &= \pm q^{\mp 1/4}\left(q^{1/4}X_{12}^{\pm}X_3^{\pm}-q^{-1/4}X_3^{\pm}X_{12}^{\pm}\right). \end{aligned} \tag{2b}$$

All other commutation relations for the generators follow from these definitions [1] $(X_{ii}^{\pm}\equiv X_i^{\pm})$:

$$[X_a^{+}, X_{ab}^{-}]=-q^{H_a/2}X_{a+1b}^{-} \qquad [X_b^{+},\ X_{ab}^{-}]=X_{ab-1}^{-}q^{-H_b/2} \qquad 1\leqslant a<b\leqslant 3 \tag{3a}$$

$$[X_a^{-}, X_{ab}^{+}]=X_{a+1b}^{+}q^{-H_a/2} \qquad [X_b^{-}, X_{ab}^{+}]=-q^{H_b/2}X_{ab-1}^{+} \qquad 1\leqslant a<b\leqslant 3 \tag{3b}$$

$$X_a^{\pm}X_{ab}^{\pm}=q^{1/2}X_{ab}^{\pm}X_a^{\pm} \qquad X_b^{\pm}X_{ab}^{\pm}=q^{-1/2}X_{ab}^{\pm}X_b^{\pm} \qquad 1\leqslant a<b\leqslant 3 \tag{3c}$$

$$X_{12}^{\pm}X_{13}^{\pm}=q^{1/2}X_{13}^{\pm}X_{12}^{\pm} \qquad X_{23}^{\pm}X_{13}^{\pm}=q^{-1/2}X_{13}^{\pm}X_{23}^{\pm} \tag{3d}$$

$$[X_2^{\pm} \qquad X_{13}^{\pm}]=0 \qquad [X_2^{\pm}, X_{13}^{\mp}]=0 \tag{3e}$$

$$[X_{12}^{+}, X_{13}^{-}]=-q^{H_1+H_2}X_3^{-} \qquad [X_{12}^{-}, X_{13}^{+}]=X_3^{+}q^{-H_1-H_2} \tag{3f}$$

$$[X_{23}^{+}, X_{13}^{-}]=X_1^{-}q^{-H_2-H_3} \qquad [X_{23}^{-}, X_{13}^{+}]=-q^{H_2+H_3}X_1^{+} \tag{3g}$$

$$[X_{12}^{\pm}, X_{23}^{\pm}]=\tilde{\lambda}X_2^{\pm}X_{13}^{\pm} \qquad [X_{12}^{\pm}, X_{23}^{\mp}]=-\tilde{\lambda}q^{\pm H_2/2}X_1^{\pm}X_3^{\mp}. \tag{3h}$$

We shall not need the coalgebra structure of $\mathcal{U}_q(sl(4,\mathbb{C}))$ [3,4] and $\mathcal{U}_q(su(2,2))$ [1] and thus it will not be given here.

3. The generic case

We first consider the representations of $\mathcal{U}_q(su(2,2))$ when the deformation parameter is not a root of unity. In this case the representations of $\mathcal{U}_q(su(2,2))$ we use are irreducible lowest-weight modules M^{Λ} (in particular, Verma modules V^{Λ}) of $\mathcal{U}_q(sl(4,\mathbb{C}))$, together with the reality condition necessary for the construction of the scalar product in M^{Λ} (or V^{Λ}). The module M^{Λ} is given by its lowest weight $\Lambda\in\mathcal{H}^*$ ($\mathcal{H}^*$ being the dual of $\mathcal{H}$) and lowest-weight vector $v_0\equiv v_0(\Lambda)$, such that v_0 is annihilated by the lowering generators, $Xv_0=0$, $X\in\mathcal{U}_q(\mathcal{G}^-)$ and $Hv_0=\Lambda(H)v_0$ for any Cartan generator H.

In particular, we use the Verma modules V^Λ which are the lowest-weight modules such that $V^\Lambda = \mathcal{U}_q(\mathcal{G}^+)\,v_0$. Thus the Poincaré–Birkhof–Witt theorem tells us that the basis of V^Λ consists of monomial vectors

$$\Psi_{\{\bar{k}\}} = (X_{13}^+)^{k_{13}}(X_{12}^+)^{k_{12}}(X_{23}^+)^{k_{23}}(X_2^+)^{k_2}(X_1^+)^{k_1}(X_3^+)^{k_3}v_0 = \mathcal{P}_{\{\bar{k}\}}v_0 \qquad k_j, k_{jk} \in \mathbb{Z}_+. \tag{4}$$

In order to consider V^Λ as a representation of the real form we need, as in the $q = 1$ case, a Hermiticity condition invariant with respect to $\mathcal{U}_q(su(2,2))$. This is given by [2]:

$$\omega(H) = H \qquad \forall H \in \mathcal{H} \qquad \omega(X_{jk}^\pm) = \begin{cases} X_{jk}^\mp & (jk) = (11), (33) \\ -X_{jk}^\mp & \text{otherwise} \end{cases} \tag{5}$$

where ω acts as $\mathbb{C}$-anti-linear algebra anti-involution of $\mathcal{U}_q(sl(4,\mathbb{C}))$. Using the conjugation ω, the following $\mathcal{U}_q(su(2,2))$-invariant scalar product can be defined [2]:

$$\left(\Psi_{\{\bar{k}'\}}, \Psi_{\{\bar{k}\}}\right) = \left(\mathcal{P}_{\{\bar{k}'\}}v_0, \mathcal{P}_{\{\bar{k}\}}v_0\right) = \left(v_0, \omega\left(\mathcal{P}_{\{\bar{k}'\}}\right)\mathcal{P}_{\{\bar{k}\}}v_0\right) \tag{6}$$

with $(v_0, v_0) = 1$.

Generically, the Verma modules V^Λ are irreducible. A Verma module V^Λ is reducible [5] iff there exists a positive root α, $\alpha = \sum_k n_k\alpha_k$, $n_k \in \mathbb{Z}_+$, α_k being the simple roots, and a positive integer m_α such that the following equality holds:

$$\left[(\Lambda - \rho)(H_\alpha) + m_\alpha\right]_q = 0 \tag{7}$$

where $H_\alpha = \sum_k n_k H_k$, and ρ is half the sum of the positive roots; note that $\rho(H_k) = 1$. For the six positive roots of the root system of $sl(4,\mathbb{C})$, one has (see [6]):

$$m_1 = -\Lambda(H_1) + 1 = 2j_1 + 1 \tag{8a}$$
$$m_2 = -\Lambda(H_2) + 1 = 1 - d - j_1 - j_2 \tag{8b}$$
$$m_3 = -\Lambda(H_3) + 1 = 2j_2 + 1 \tag{8c}$$
$$m_{12} = -\Lambda(H_{12}) + 2 = m_1 + m_2 = 2 - d + j_1 - j_2 \tag{8d}$$
$$m_{23} = -\Lambda(H_{23}) + 2 = m_2 + m_3 = 2 - d - j_1 + j_2 \tag{8e}$$
$$m_{13} = -\Lambda(H_{13}) + 3 = m_1 + m_2 + m_3 = 3 - d + j_1 + j_2 \tag{8f}$$

where we use the classical labelling of the $su(2,2)$ representations: d is the conformal dimension (or energy) and $2j_1, 2j_2$ are non-negative integers fixing finite-dimensional irreducible representations of the $U_q(su(2)) \otimes U_q(su(2))$ subalgebra. The latter integers fix also the finite-dimensional irreducible representations of a q-Lorentz algebra, though it is a subalgebra of this q-conformal algebra only for $q = 1$.

Whenever (7) is satisfied, there exists a singular (null) vector $v_s^{m,\alpha}$ in V^Λ such that $v_s^{m,\alpha} \neq v_0$, $Xv_s^{m,\alpha} = 0$, $\forall X \in \mathcal{U}_q(\mathcal{G}^-)$ and $Hv_s^{m,\alpha} = (\Lambda + m\alpha)(H)\,v_s^{m,\alpha}$, $\forall H \in \mathcal{H}$. The space $I^{m,\alpha} = U_q(\mathcal{G}^+)\,v_s^{m,\alpha}$ is a proper submodule of V^Λ isomorphic to the Verma module $V^{\Lambda+m\alpha}$ with a shifted lowest weight $\Lambda + m\alpha$ [5].

The Verma module V^Λ contains a unique proper maximal submodule I^Λ (which contains all submodules $I^{m,\alpha}$). Among the lowest-weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e. $L_\Lambda = V^\Lambda/I^\Lambda$. (If V^Λ is irreducible then $L_\Lambda = V^\Lambda$.) To obtain the irreducible lowest-weight module L_Λ we have to factor out all singular vectors. First of all, we have that m_1 and m_3 are positive, since $2j_1$ and $2j_2$ are non-negative integers. The corresponding singular vectors are

$$v_1 = \left(X_1^+\right)^{2j_1+1} v_0 \qquad v_3 = \left(X_3^+\right)^{2j_2+1} v_0 \tag{9}$$

and these are present for all representations we discuss.

Further, we concentrate on the *massless* case [2, 7], for which

$$d = j_1 + j_2 + 1 \qquad j_1 j_2 = 0. \tag{10}$$

For definiteness we choose $j_2 = 0$. Then we see that in the case $j_1 \neq 0$ we have two more singular vectors corresponding to $m_{12} = 1$ and $m_{13} = 2$ [2]:

$$v_{12} = \left([2j_1]X_{12}^+ - q^{j_1} X_2^+ X_1^+\right) v_0 \qquad d = j_1 + 1 \qquad j_2 = 0 \qquad m_{12} = 1 \tag{11}$$

$$\begin{aligned} v_{13}^{(2)} = X_3^+ \big([2j_1][2j_1 - 1](X_1^+)^2 (X_2^+)^2 - [2][2j_1 + 1][2j_1 - 1] X_1^+ (X_2^+)^2 X_1^+ \\ + [2j_1 + 1][2j_1](X_2^+)^2 (X_1^+)^2\big) X_3^+ v_0 \\ d = j_1 + 1 \qquad j_2 = 0 \qquad m_{13} = 2. \end{aligned} \tag{12}$$

Note however, that the singular vector $v_{13}^{(2)}$ is a composite one. It gives no new condition since, factoring out the submodule I_3 generated by the singular vector $v_3 = X_3^+ v_0$, we factor out also the submodule $I_{13}^{(2)}$ generated by $v_{13}^{(2)}$; $I_{13}^{(2)}$ is a submodule of I_3.

When $j_1 = 0$, besides (11) and (12) (with $j_1 = 0$), there is another singular vector corresponding to $m_{23} = 1$ [2]:

$$v_{23} = X_2^+ X_3^+ v_0 \qquad d = 1 \qquad j_1 = j_2 = 0 \qquad m_{23} = 1 \tag{13}$$

which is also composite. Furthermore, for $j_1 = 0$ the vector $v_{12} = X_2^+ X_1^+ v_0$ is also composite, since in this case the submodule I_{12} is also a submodule of the submodule I_1 generated by $v_1 = X_1^+ v_0$.

Let us factor out all above singular vectors. This means that in the factor representation, whose ground-state vector we denote by $\widetilde{|\rangle}$, we have:

$$\left(X_1^+\right)^{2j_1+1} \widetilde{|\rangle} = 0 \tag{14a}$$

$$X_3^+ \widetilde{|\rangle} = 0 \tag{14b}$$

$$\left([2j_1]X_{12}^+ - q^{j_1} X_2^+ X_1^+\right) \widetilde{|\rangle} = 0. \tag{14c}$$

In the classical case this factor representation is still reducible [8]. Also, there is an additional singular vector here:

$$v_f = \left(X_{13}^+ X_2^+ - q^{-1/2} X_{12}^+ X_{23}^+\right) \widetilde{|\rangle}. \tag{15}$$

Checking singularity of v_f, i.e. $X_a^- v_f = 0$, we see that it is straightforward for $a = 1, 3$, while for $a = 2$ we have to use (14*b*) and, if $j_1 \neq 0$, also (14*c*). Factoring out the submodule built on v_f we obtain the irreducible lowest-weight representation L_Λ whose vacuum vector $|\rangle$ obeys the equations

$$\left(X_1^+\right)^{2j_1+1} |\rangle = 0 \tag{16a}$$

$$X_3^+ |\rangle = 0 \tag{16b}$$

$$\left([2j_1]X_{12}^+ - q^{j_1} X_2^+ X_1^+\right) |\rangle = 0 \tag{16c}$$

$$\left(X_{13}^+ X_2^+ - q^{-1/2} X_{12}^+ X_{23}^+\right) |\rangle = 0. \tag{16d}$$

As we noted above for $j_1 = 0$, (16c) follows from (16a). We also note that

$$([2j_1]X_{13}^+ - q^{j_1}X_{23}^+X_1^+)|\,\rangle = 0 \tag{17}$$

which is a simple consequence of (16b, c).†

We can now explicitly give the basis of L_Λ. *We consider the monomials as in* (4), *but* on the vacuum $|\,\rangle$, i.e.

$$\Phi_{\{\bar{k}\}} = (X_{13}^+)^{k_{13}}(X_{12}^+)^{k_{12}}(X_{23}^+)^{k_{23}}(X_2^+)^{k_2}(X_1^+)^{k_1}(X_3^+)^{k_3}|\,\rangle = \mathcal{P}_{\{\bar{k}\}}|\,\rangle \qquad k_j, k_{jk} \in \mathbb{Z}_+. \tag{18}$$

First of all we have the restrictions $k_1 \leqslant 2j_1$ and $k_3 = 0$ because of (16a, b). Then we have $k_1k_2 = 0$ because of (16c), since any occurrence of $X_2^+X_1^+$ is replaced by X_{12}^+. In the same way we have $k_{12}k_{23} = 0$ because of (16d). Finally, $k_1k_{23} = 0$ because of (17).

Thus, the basis of L_Λ consists of the monomials

$$\Phi^1_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{12}^+)^\ell(X_2^+)^n|\,\rangle \qquad k, \ell, n \in \mathbb{Z}_+ \tag{19a}$$

$$\Phi^2_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{23}^+)^\ell(X_2^+)^n|\,\rangle \qquad k, n \in \mathbb{Z}_+ \qquad \ell \in \mathbb{N} \tag{19b}$$

$$\Phi^3_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{12}^+)^\ell(X_1^+)^n|\,\rangle \qquad k, \ell, n \in \mathbb{Z}_+ \qquad 1 \leqslant n \leqslant 2j_1 \tag{19c}$$

the third case being absent for $j_1 = 0$; $\ell \neq 0$ in (19b) to avoid coincidence with (19a) for $\ell = 0$.

Note that the different vectors in (19) have different weights. Thus each weight has multiplicity one and is represented by a single vector. The norms squared of the basis vectors,

$$\|\Phi^a_{\{k,\ell,n\}}\|^2 \equiv \left(\Phi^a_{\{k,\ell,n\}}, \Phi^a_{\{k,\ell,n\}}\right) \tag{20}$$

are given explicitly by

$$\|\Phi^1_{\{k,\ell,n\}}\|^2[k]_q![k+\ell]_q!\ [\ell+n]_q![n+2j_1]_q!/[2j_1]_q! \tag{21a}$$

$$\|\Phi^2_{\{k,\ell,n\}}\|^2 = [k]_q![k+\ell]_q!\ [\ell+n+2j_1]_q![n]_q!/[2j_1]_q! \tag{21b}$$

$$\|\Phi^3_{\{k,\ell,n\}}\|^2 = [k]_q![k+\ell+n]_q!\ [\ell]_q![2j_1]_q!/[2j_1-n]_q! \tag{21c}$$

where $[x]_q! \equiv [x]_q[x-1]_q\ldots[1]_q$ for $x \in \mathbb{N}$, $[0]_q! \equiv 1$. For the rest of this section we suppose that q is not a non-trivial root of unity. These norms are non-vanishing, but they can be negative for $q \neq 1$ and large enough k, l and n. The corresponding representations are then pseudo-unitary and become unitary only in the limit $q \to 1$, when all norms are strictly positive.

It will be also convenient to work with the normalized basis

$$\hat{\Phi}^a_{\{k,\ell,n\}} = \frac{\Phi^a_{\{k,\ell,n\}}}{\|\Phi^a_{\{k,\ell,n\}}\|} \qquad a = 1, 2, 3 \tag{22}$$

even when the norms squared are negative; this will simplify the treatment of the case of q being a root of unity (see next section). The vectors (22) are in fact pseudo-orthonormal (orthonormal for $q = 1$) since, as we noted above, they have different weights and one has

$$\left(\hat{\Phi}^a_{\{k,\ell,n\}}, \hat{\Phi}^b_{\{k',\ell',n'\}}\right) = \varepsilon^a_{k\ell n}\delta_{ab}\delta_{kk'}\delta_{\ell\ell'}\delta_{nn'} \qquad \varepsilon^a_{k\ell n} = \mathrm{sign}\|\Phi^a_{\{k,\ell,n\}}\|^2. \tag{23}$$

† We also use $([2j_1+1]X_{12}^+ - q^{j_1}X_1^+X_2^+)|\,\rangle = 0$, $(X_{13}^+X_2^+ - q^{1/2}X_{23}^+X_{12}^+)|\,\rangle = 0$ and $([2j_1+1]X_{13}^+ - q^{j_1}X_1^+X_{23}^+)|\,\rangle = 0$, which are equivalent to (16c), (16d) and (17), respectively.

The transformation rules for this basis can be computed explicitly:

$$X_1^- \hat\Phi^1_{\{k,\ell,n\}} = q^{(2j_1+n+1-k-\ell)/2}\sqrt{[k+\ell]_q[n+2j_1+1]_q}\begin{cases}\hat\Phi^1_{\{k,\ell-1,n+1\}} & \ell>0\\ \hat\Phi^2_{\{k-1,1,n\}} & \ell=0\end{cases} \tag{24a}$$

$$X_1^- \hat\Phi^2_{\{k,\ell,n\}} = q^{(2j_1+n+1-k+\ell)/2}\sqrt{[k]_q[\ell+n+2j_1+1]_q}\,\hat\Phi^2_{\{k-1,\ell+1,n\}} \tag{24b}$$

$$X_1^- \hat\Phi^3_{\{k,\ell,n\}} = q^{-(k+\ell)/2}\sqrt{[k+\ell+n]_q[2j_1+1-n]_q}\,\hat\Phi^3_{\{k,\ell,n-1\}} \tag{24c}$$

$$X_2^- \hat\Phi^1_{\{k,\ell,n\}} = -q^{\ell/2}\sqrt{[\ell+n]_q[2j_1+n]_q}\begin{cases}\hat\Phi^1_{\{k,\ell,n-1\}} & n>0\\ q^{j_1-\frac12}\hat\Phi^3_{\{k,\ell-1,1\}} & n=0\end{cases} \tag{25a}$$

$$X_2^- \hat\Phi^2_{\{k,\ell,n\}} = -q^{-\ell/2}\sqrt{[n]_q[2j_1+\ell+n]_q}\,\hat\Phi^2_{\{k,\ell,n-1\}} \tag{25b}$$

$$X_2^- \hat\Phi^3_{\{k,\ell,n\}} = -q^{(2j_1+\ell-n-1)/2}\sqrt{[\ell]_q[2j_1-n]_q}\,\hat\Phi^3_{\{k,\ell-1,n+1\}} \tag{25c}$$

$$X_3^- \hat\Phi^1_{\{k,\ell,n\}} = -q^{(k-\ell-n-2)/2}\sqrt{[k]_q[\ell+n+1]}\,\hat\Phi^1_{\{k-1,\ell+1,n\}} \tag{26a}$$

$$X_3^- \hat\Phi^2_{\{k,\ell,n\}} = -q^{(k+\ell-n-2)/2}\sqrt{[k+\ell]_q[n+1]_q}\,\hat\Phi^2_{\{k,\ell-1,n+1\}} \tag{26b}$$

$$X_3^- \hat\Phi^3_{\{k,\ell,n\}} = -q^{(k-\ell-2)/2}\sqrt{[k]_q[\ell+1]_q}\,\hat\Phi^3_{\{k-1,\ell+1,n\}} \tag{26c}$$

$$H_1\hat\Phi^1_{\{k,\ell,n\}} = (k+\ell-n-2j_1)\hat\Phi^1_{\{k,\ell,n\}} \tag{27a}$$

$$H_1\hat\Phi^2_{\{k,\ell,n\}} = (k-\ell-n-2j_1)\hat\Phi^2_{\{k,\ell,n\}} \tag{27b}$$

$$H_1\hat\Phi^3_{\{k,\ell,n\}} = (k+\ell+2n-2j_1)\hat\Phi^3_{\{k,\ell,n\}} \tag{27c}$$

$$H_2\hat\Phi^1_{\{k,\ell,n\}} = (\ell+2n+2j_1+1)\hat\Phi^1_{\{k,\ell,n\}} \tag{28a}$$

$$H_2\hat\Phi^2_{\{k,\ell,n\}} = (\ell+2n+2j_1+1)\hat\Phi^2_{\{k,\ell,n\}} \tag{28b}$$

$$H_2\hat\Phi^3_{\{k,\ell,n\}} = (\ell-n+2j_1+1)\hat\Phi^3_{\{k,\ell,n\}} \tag{28c}$$

$$H_3\hat\Phi^1_{\{k,\ell,n\}} = (k-\ell-n)\hat\Phi^1_{\{k,\ell,n\}} \tag{29a}$$

$$H_3\hat\Phi^2_{\{k,\ell,n\}} = (k+\ell-n)\hat\Phi^2_{\{k,\ell,n\}} \tag{29b}$$

$$H_3\hat\Phi^3_{\{k,\ell,n\}} = (k-\ell)\hat\Phi^3_{\{k,\ell,n\}} \tag{29c}$$

$$X_1^+ \hat\Phi^1_{\{k,\ell,n\}} = q^{(k+\ell-n)/2}\sqrt{[k+\ell+1]_q[2j_1+n]_q}\begin{cases}q^{\frac12-j_1}\hat\Phi^1_{\{k,\ell+1,n-1\}} & n>0\\ \hat\Phi^3_{\{k,\ell,1\}} & n=0\end{cases} \tag{30a}$$

$$X_1^+ \hat\Phi^2_{\{k,\ell,n\}} = q^{(-2j_1+k-\ell-n+1)/2}\sqrt{[k+1]_q[2j_1+\ell+n]_q}\,\hat\Phi^2_{\{k+1,\ell-1,n\}} \tag{30b}$$

$$X_1^+ \hat\Phi^3_{\{k,\ell,n\}} = q^{(k+\ell)/2}\sqrt{[k+\ell+n+1]_q[2j_1-n]_q}\,\hat\Phi^3_{\{k,\ell,n+1\}} \tag{30c}$$

$$X_2^+ \hat\Phi^1_{\{k,\ell,n\}} = q^{-\ell/2}\sqrt{[\ell+n+1]_q[2j_1+1+n]_q}\,\hat\Phi^1_{\{k,\ell,n+1\}} \tag{31a}$$

$$X_2^+ \hat\Phi^2_{\{k,\ell,n\}} = q^{\ell/2}\sqrt{[2j_1+1+\ell+n]_q[n+1]_q}\,\hat\Phi^2_{\{k,\ell,n+1\}} \tag{31b}$$

$$X_2^+ \hat\Phi^3_{\{k,\ell,n\}} = q^{(-2j_1+n-\ell-1)/2}\sqrt{[\ell+1]_q[2j_1+1-n]_q}\,\hat\Phi^3_{\{k,\ell+1,n-1\}} \tag{31c}$$

$$X_3^+ \hat\Phi^1_{\{k,\ell,n\}} = -q^{(\ell+n-k)/2}\sqrt{[k+1]_q[\ell+n]_q}\begin{cases}\hat\Phi^1_{\{k+1,\ell-1,n\}} & \ell>0\\ \hat\Phi^2_{\{k,1,n-1\}} & \ell=0\end{cases} \tag{32a}$$

$$X_3^+ \hat\Phi^2_{\{k,\ell,n\}} = -q^{(n-k-\ell)/2}\sqrt{[k+\ell+1]_q[n]_q}\,\hat\Phi^2_{\{k,\ell+1,n-1\}} \tag{32b}$$

$$X_3^+ \hat\Phi^3_{\{k,\ell,n\}} = -q^{(\ell-k)/2}\sqrt{[k+1]_q[\ell]_q}\,\hat\Phi^3_{\{k+1,\ell-1,n\}}\,. \tag{32c}$$

We have thus proved the following theorem:

Theorem 1. The basis vectors $\Phi^a_{\{k,\ell,n\}}$ given in (19) span a representation space for the pseudo-unitary massless irreducible representation of $\mathcal{U}_q(su(2,2))$ with $d = j_1 + 1$, $j_2 = 0$, $|q| = 1$, q not a non-trivial root of unity.

Remark. Note that such a basis is also new for the algebra $su(2,2)$, i.e. for $q = 1$. As already remarked, in this case the representations become unitary. Since the weight spaces are one-dimensional, one may call the massless irreducible representatins *singletons* using terminology of [9].

We would also like to give an interpretation of the representation basis via character formulae. Such formulae represent the basis vectors through formal variables, $t_j = e(\alpha_j)$, $j = 1, 2, 3$ which correspond to the simple root vectors X_j^+, and $e()$ have formal properties of the exponential function, namely, $e(\mu)e(\nu) = e(\mu+\nu)$, $e(0) = 1$. Thus, the non-simple root vectors X_{12}^+, X_{23}^+, X_{13}^+ are represented by $t_{12} = e(\alpha_{12}) = t_1t_2$, $t_{23} = e(\alpha_{23}) = t_2t_3$, $t_{13} = e(\alpha_{13}) = t_1t_2t_3$. When q is not a root of 1, the massless irreps can be represented by the following character formulae (containing the same information as (19):

$$\operatorname{ch} L = e(\Lambda)\Big(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} t_{13}^k t_{12}^\ell t_2^n + \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} t_{13}^k t_{23}^\ell t_2^n + \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{n=1}^{2j_1} t_{13}^k t_{12}^\ell t_1^n\Big) \tag{33}$$

where the overall prefactor $e(\Lambda)$ represents the lowest-weight state.

In the same fashion the character formula for the Verma module with the same lowest weight is (cf e.g. [10])

$$\operatorname{ch} V^\Lambda = e(\Lambda)/(1-t_1)(1-t_2)(1-t_3)(1-t_{12})(1-t_{23})(1-t_{13}). \tag{34}$$

which has the same content as (4) with $k_j, k_{j\ell} \in \mathbb{Z}_+$.

Now we can rewrite the character formula (33) as

$$\begin{aligned}
\operatorname{ch} L_\Lambda &= \operatorname{ch} V^\Lambda Q(t_1,t_2,t_3)\\
&= \operatorname{ch} V^\Lambda\big(1 - t_1^{m_1} + t_1^{m_1}t_3 - t_3 - t_1t_2 + t_1^{m_1}t_2 - t_1^{m_1}t_2t_3^2 + t_1t_2t_3^2\\
&\qquad - t_1^{m_1}t_2^2t_3 + t_1^2t_2^2t_3 - t_1^2t_2^2t_3^2 + t_1^{m_1}t_2^2t_3^2\big)\\
m_1 &= 2j_1 + 1 \geqslant 1 \qquad d = j_1 + 1 \qquad j_2 = 0.
\end{aligned} \tag{35}$$

This formula is valid for all $j_1 \in \frac{1}{2}\mathbb{Z}_+$, $j_2 = 0$. Note, however, that for $j_1 = \frac{1}{2}$ the terms in the fourth row cancel each other out, while for $j_1 = 0$ the terms in the third row cancel each other. To show that (35) concides with (33) amounts to the explicit straightforward division of the polynomials:

$$\frac{Q(t_1,t_2,t_3)}{(1-t_1)(1-t_2)(1-t_3)(1-t_{12})(1-t_{23})(1-t_{13})}. \tag{36}$$

Formula (35) represents an alternating sign summation over a part of the Weyl group of $sl(4,\mathbb{C})$ and may be obtained using [11, 12]. Note, however, that the ultimate formula is (33) which we obtained in a straightforward manner.

Finally we mention the necessary changes in order to treat the massless case with $d = j_2 + 1$, $j_1 = 0$. Instead of (16) we have:

$$X_1^+|\rangle = 0 \tag{37a}$$
$$(X_3^+)^{2j_2+1}|\rangle = 0 \tag{37b}$$
$$\left([2j_2]X_{23}^+ + q^{-j_2-\frac{1}{2}}X_2^+X_3^+\right)|\rangle = 0 \tag{37c}$$
$$\left(X_{13}^+X_2^+ - q^{1/2}X_{23}^+X_{12}^+\right)|\rangle = 0. \tag{37d}$$

Instead of (19) the basis is given by

$$\Phi'^1_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{23}^+)^\ell(X_2^+)^n|\rangle \qquad k,\ell,n \in \mathbb{Z}_+ \tag{38a}$$
$$\Phi'^2_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{12}^+)^\ell(X_2^+)^n|\rangle \qquad k,n \in \mathbb{Z}_+,\ \ell \in \mathbb{N} \tag{38b}$$
$$\Phi'^3_{\{k,\ell,n\}} = (X_{13}^+)^k(X_{23}^+)^\ell(X_3^+)^n|\rangle \qquad k,\ell,n \in \mathbb{Z}_+,\ 1 \leqslant n \leqslant 2j_2. \tag{38c}$$

The character formula for $j_2 \in \frac{1}{2}\mathbb{Z}_+$, $j_1 = 0$ is obtained from (35) by changing $t_1 \leftrightarrow t_3$, $m_1 \to m_3$, $j_1 \leftrightarrow j_2$.

4. The case q a root of unity

We now turn to the case of the deformation parameter q being a root of unity, namely, $q = e^{2\pi i/N}$, $N = 2, 3, \ldots$.

In this situation all Verma modules V^Λ are reducible [5] and all irreducible representations are *finite-dimensional* [13]. There are now more singular vectors; these were given in [2]. Instead of working with them, we can consider directly the representation space constructed before and find explicitly how it is reduced in the present case.

We consider first the norms given in (20). Using the fact that $[sN]_q = 0$, $\forall s \in \mathbb{Z}$ we see that the following norms vanish:

$$\|\Phi^1_{\{k,\ell,n\}}\|^2 = 0 \qquad \text{for}\quad k+\ell \geqslant N \quad\text{or}\quad \ell+n \geqslant N \quad\text{or}\quad n \geqslant N-2J_1 \tag{39a}$$
$$\|\Phi^2_{\{k,\ell,n\}}\|^2 = 0 \qquad \text{for}\quad k+\ell \geqslant N \quad\text{or}\quad \ell+n \geqslant N-2J_1 \tag{39b}$$
$$\|\Phi^3_{\{k,\ell,n\}}\|^2 = 0 \qquad \text{for}\quad k+\ell+n \geqslant N \tag{39c}$$

where we have decomposed $2j_1 = 2J_1 + sN$, $0 \leqslant 2J_1 < N$, $J_1 \in \frac{1}{2}\mathbb{Z}_+$, $s \in \mathbb{Z}_+$. Then we note that

$$[p+2j_1]_q = [p+2J_1+sN]_q = (-1)^s[p+2J_1]_q \tag{40a}$$
$$[p+2j_1]_q!/[2j_1]_q! = (-1)^{ps}[p+2J_1]_q!/[2J_1]_q! \tag{40b}$$

and applying this to (21) we obtain

$$\|\Phi^1_{\{k,\ell,n\}}\|^2 = (-1)^{sn}[k]_q![k+\ell]_q![\ell+n]_q![n+2J_1]_q!/[2J_1]_q! \tag{41a}$$
$$\|\Phi^2_{\{k,\ell,n\}}\|^2 = (-1)^{s(\ell+n)}[k]_q![k+\ell]_q![\ell+n+2J_1]_q![n]_q!/[2J_1]_q! \tag{41b}$$
$$\|\Phi^3_{\{k,\ell,n\}}\|^2 = (-1)^{sn}[k]_q![k+\ell+n]_q![\ell]_q![2J_1]_q!/[2J_1-n]_q!\,. \tag{41c}$$

Obviously the above norms can be positive for all k, ℓ, n only if $s = 2r$, $r \in \mathbb{Z}_+$. Thus, we recover the result announced in [2] that the finite-dimensional massless irreducible representations for roots of unity are unitary iff

$$d = j_1 + 1 \qquad j_2 = 0 \qquad 2rN \leqslant 2j_1 \leqslant (2r+1)N - 1 \qquad \forall r \in \mathbb{Z}_+. \tag{42}$$

For fixed j_1 in the above range, the basis of the unitary irreducible representation is given by

$$\Phi^1_{\{k,\ell,n\}} \qquad k, \ell, n \in \mathbb{Z}_+ \quad k+\ell, \ell+n \leqslant N-1 \quad n \leqslant N - 2J_1 - 1 \tag{43a}$$

$$\Phi^2_{\{k,\ell,n\}} \qquad k, n \in \mathbb{Z}_+ \quad \ell \in \mathbb{N} \quad k+\ell \leqslant N-1 \quad \ell+n \leqslant N - 2J_1 - 1 \tag{43b}$$

$$\Phi^3_{\{k,\ell,n\}} \qquad k, \ell, n \in \mathbb{Z}_+ \quad k+\ell+n \leqslant N-1 \quad 1 \leqslant n \leqslant 2J_1 \tag{43c}$$

where $j_1 = J_1 + rN$, $0 \leqslant 2J_1 < N$, $r \in \mathbb{Z}_+$. The norms of these vectors are given by (21) with j_1 replaced by J_1.

Analogously to section 2, we introduce also the orthonormal basis $\hat{\Phi}^a_{\{k,\ell,n\}}$ for which we have the same transformation laws (24)–(32), with j_1 replaced by J_1. For consistency we have to check that the basis given in (43) is indeed a representation space. It is enough to consider the boundary cases, i.e. the cases in which acting on a vector in (43) would result in a vector not in (43). However, we observe simply by inspection that in all such cases the coefficient on the RHS of the corresponding formula in (24)–(32) is zero. (For example, $X_1^- \hat{\Phi}^1_{\{k,\ell,N-2J_1-1\}} = 0 \cdot \hat{\Phi}^1_{\{k,\ell-1,N-2J_1\}}$ $(\ell > 0)$.)

Thus, we have proved the following theorem:

Theorem 2. The basis vectors $\Phi^a_{\{k,\ell,n\}}$ given in (43) span a representation space for the unitary massless irreducible representation of $\mathcal{U}_q(su(2,2))$ with $d = j_1 + 1$, $j_2 = 0$, and $q = e^{2\pi i/N}$.

Having established the basis, we can count the number of states in it. We find that the number of states in (43*a*), (43*b*), (43*c*) is, respectively,

$$\tfrac{1}{6}(N-2J_1)(2N^2 + N(4J_1+3) + 1 - 4J_1^2) \tag{44a}$$

$$\tfrac{1}{6}(N-2J_1)(N-2J_1-1)(2N+2J_1-1) \tag{44b}$$

$$\tfrac{1}{3}J_1(3N^2 - 6NJ_1 - 1 + 4J_1^2). \tag{44c}$$

The sum of these three numbers gives the dimension of the massless irreps, which was derived in [2] from other considerations:

$$d(N, J_1) = \tfrac{1}{3}[2N^3 - N(12J_1^2 - 1) + 3J_1(4J_1^2 - 1)]. \tag{45}$$

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